2. V. V. Pukhnachev, "Development of an anomalous thermocapillary effect in a thin layer of liquid," in: Hydrodynamics and Heat Exchange of Liquid Flows with a Free Surface [in Russian], IT SO Akad. Nauk SSSR, Novosibirsk (1985).
3. Yu. S. Ryazantsev, V. M. Shevtsova, and Yu. A. Val'tsiferov, "Effect of form of temperature dependence of surface tension on movement and heat exchange in a layer of liquid with local heating," Inzh.-Fiz. Zh., 57, No. 1 (1989).
4. R. Vochten and G. Petre, "Study of the heat of reversible adsorption at the air—solution interface. II. Experimental determination of the heat of reversible adsorption of some alcohols," J. Colloid Int. Sci., 42, No. 2 (1973).
5. J. C. Legros, M. C. Limbourg, and G. Petre, "Influence of a surface tension minimum as a function of temperature on the Marangoni convection," Acta Astronaut., 11, No. 2 (1984).
6. V. A. Al'vares-Suares and Yu. S. Ryazantsev, "Thermocapillary movement caused by local heating of liquid by a pulse of ultraviolet radiation," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1986).
7. L. D. Landau and E. M. Lifshits, Hydrodynamics [in Russian], Nauka, Moscow (1986).
8. Yu. V. Val'tsiferov, Yu. S. Ryazantsev, and V. M. Shevtsova, "Study of thermocapillary and thermogravitation convection in a liquid with local heating," Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, No. 6 (1989).
9. S. F. Rastopov and A. T. Sukhodol'skii, "Use of the laser-induced Marangoni effect for recording diffraction grids," Kvantovaya Élektron., 14, No. 8 (1987).

ANALOG OF THE SHALLOW-WATER VORTEX EQUATION FOR HOLLOW AND TORNADO-LIKE VORTICES.

HEIGHT OF A STEADY TORNADO-LIKE VORTEX

V. V. Nikulin

UDC 532.5

The analog of the shallow-water vortex equation for hollow and tornado-like vortices is obtained in the long-wave approximation for an inviscid, incompressible, and nonuniform fluid. A steady, vertical tornado-like vortex is examined, whose central fluid is lighter than that outside the center. A sharp criterion is obtained, which distinguishes the case where the flow is bounded or unbounded in height. Calculation of the vortex height according to theoretical formula agrees in order of magnitude with the results of laboratory measurements and observations of naturally occurring dust devils.

1. Let us consider an incompressible, inviscid, nonuniform fluid in a gravitational field. The flow is assumed to be rotationally symmetric. We introduce a cylindrical coordinate system (r, φ, z), where r is the radius, and φ is the aximuthal angle. The z axis is directed opposite the force of gravity. The flow is divided into two regions in space: in region I, $r \le r_0(z, t)$; in region II, $r_0(z, t) \le r \le r_*$. Here r_* is a constant, r_0 is in general a function of z and t, and t is the time. At the boundary r_0, there can be a discontinuity in density and the component of velocity tangential to this boundary. The velocity components corresponding to (r, φ, z) are denoted by (u, v, w), and p, ρ, g are the pressure, density, and acceleration of gravity, respectively.

In order to change over to the long-wave approximation, subsequently we introduce characteristic length, velocity, and density scales. As the unit of length, we adopt the characteristic scale of change along the z axis and for unit velocity, the magnitude of the rotational component for $r = r_0$, $z = 0$, $t = 0$. The characteristic density is set equal to 1. Then the characteristic time, pressure, and acceleration are equal to 1. The characteristic scale for change along the r axis is denoted by δ. It is assumed that $\delta \ll 1$.

Novosibirsk. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 2, pp. 47-52, March-April, 1992. Original article submitted December 3, 1990.

Transformation to new variables and functions is accomplished by the use of

$$r^2 \to \delta^2\eta,\ z \to z,\ t \to t,\ 2ur \to \delta^2 q,$$
$$vr \to \delta A,\ w \to w,\ \rho \to \rho,\ p \to p,\ g \to g.$$

The value $r = r_0$ corresponds to $\eta = \eta_0(z, t)$, $r = r_* - \eta = \eta_*$.

The equations of motion and continuity become

$$\begin{gathered}\frac{\rho\delta^2}{2}\left(\frac{\partial q}{\partial t} + q\frac{\partial q}{\partial \eta} - \frac{q^2}{2\eta} + w\frac{\partial q}{\partial z}\right) - \frac{\rho A^2}{\eta} = -2\eta\frac{\partial p}{\partial \eta},\\ \frac{\partial A}{\partial t} + q\frac{\partial A}{\partial \eta} + w\frac{\partial A}{\partial z} = 0,\\ \rho\left(\frac{\partial w}{\partial t} + q\frac{\partial w}{\partial \eta} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} - \rho g,\\ \frac{\partial q}{\partial \eta} + \frac{\partial w}{\partial z} = 0,\ \frac{\partial \rho}{\partial t} + q\frac{\partial \rho}{\partial \eta} + w\frac{\partial \rho}{\partial z} = 0.\end{gathered} \tag{1.1}$$

The following is adopted as a boundary condition:

$$q = A = 0 \text{ for } \eta = 0. \tag{1.2}$$

It is assumed that the pressure is continuous at the boundary of the region $\eta = \eta_0$, and that the kinetic condition is satisfied there:

$$q = \partial\eta_0/\partial t + w\partial\eta_0/\partial z. \tag{1.3}$$

For $\eta = \eta_*$ the impenetrability condition is satisfied

$$q = 0 \text{ for } \eta = \eta_*. \tag{1.4}$$

Furthermore, terms in (1.1) which are multiplied by δ^2 are assumed to be negligibly small, and the system is transformed by a method analogous to that proposed in vortex theory for shallow water [1]. The transformation of the equations is carried out separately for regions I and II.

We introduce new independent variables z', t', ν $(0 \le \nu \le 1)$, according to $z = z'$, $t = t'$, $\eta = R(z', t', \nu)$. Here R satisfies

$$\partial R/\partial t' + w\partial R/\partial z' = q \tag{1.5}$$

and boundary conditions

$$R(z',\ t',\ 0) = 0,\ R(z',\ t',\ 1) = \eta_0 \tag{1.6}$$

for the equations in region I and

$$R(z', t', 0) = \eta_*,\ R(z', t', 1) = \eta_0 \tag{1.7}$$

in region II. It is easy to see that the boundary conditions (1.2) (for q), (1.3), and (1.4) are automatically satisfied for this definition of R. In this case, the unknown boundary $\eta = \eta_0$ is transformed into the known value of $\nu = 1$.

For the differential operators we can write

$$\frac{\partial R}{\partial \nu}\frac{\partial}{\partial z} = \frac{\partial R}{\partial \nu}\frac{\partial}{\partial z'} - \frac{\partial R}{\partial z'}\frac{\partial}{\partial \nu},\ \frac{\partial R}{\partial \nu}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \nu},$$
$$\frac{\partial R}{\partial \nu}\frac{\partial}{\partial t} = \frac{\partial R}{\partial \nu}\frac{\partial}{\partial t'} - \frac{\partial R}{\partial t'}\frac{\partial}{\partial \nu}.$$

Using these relations and (1.5), the total derivative operator $\partial/\partial t + q\partial/\partial\eta + w\partial/\partial z$ transforms to $\partial/\partial t' + w\partial/\partial z'$. Then in the variables t', z', ν, system (1.1) takes the form (in the future, the prime on t' and z' will be omitted)

$$\frac{\rho A^2}{2R^2}\frac{\partial R}{\partial \nu}=\frac{\partial p}{\partial \nu},\ \frac{\partial A}{\partial t}+w\frac{\partial A}{\partial z}=0,$$
$$\rho\frac{\partial R}{\partial \nu}\left(\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial R}{\partial \nu}\frac{\partial p}{\partial z}+\frac{\partial R}{\partial \nu}\frac{\partial p}{\partial \nu}-\rho g\frac{\partial R}{\partial \nu},\tag{1.8}$$
$$\frac{\partial q}{\partial \nu}+\frac{\partial R}{\partial \nu}\frac{\partial w}{\partial z}-\frac{\partial R}{\partial z}\frac{\partial w}{\partial \nu}=0,\ \frac{\partial \rho}{\partial t}+w\frac{\partial \rho}{\partial z}=0.$$

It follows from (1.8) that if $A = a(\nu)$, $\rho = \rho(\nu)$ at $t = 0$, then

$$A=a(\nu),\ \rho=\rho(\nu)\tag{1.9}$$

for any t. Henceforth we will assume that (1.9) is satisfied.

An expression is obtained for p by using (1.9) and integrating the first equation in (1.8) from ν to 1. This result is substituted into the third equation in (1.8). With the help of (1.5), we eliminate q from the fourth equation and obtain a system of two equations:

$$\rho\left(\frac{\partial w}{\partial t}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p_1}{\partial z}+\frac{\rho_1 a_1^2}{2R_1^2}\frac{\partial R_1}{\partial z}+\frac{\partial}{\partial z}\int_{\nu}^{1}\frac{1}{2R}\frac{\partial(\rho a^2)}{\partial \nu}d\nu-\rho g,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial R}{\partial \nu}\right)+\frac{\partial}{\partial z}\left(w\frac{\partial R}{\partial \nu}\right)=0.\tag{1.10}$$

Here p_1, ρ_1, a_1, and R_1 are values corresponding to $\nu = 1$ (values at the boundary between regions I and II). Note that the equations have the same form (1.10) in regions I and II. The values of ρ_1 and a_1 can be different, since jumps in density and the velocity component tangential to the boundary are admissible when crossing the boundary. The values of p_1 and R_1 are the same in regions I and II, because of pressure continuity and the definition of R. We find the equations for hollow and tornado-like vortices from (1.10).

Hollow Vortex. We assume that there is no fluid in region I, and the pressure is constant and equal to zero. Then the surface $\nu = 1$ is free and $p_1 = 0$ in region II. This means that the equations for a hollow vortex have the form (1.10) with p_1 set equal to zero.

Tornado-like Vortex. We introduce another small parameter $\varepsilon = 1/\eta_*$, whose physical meaning is the ratio of the radii of regions I and II squared. The order of smallness of ε is not arbitrary. To discard terms which remain small, it follows from (1.1) that the inequality $\delta^2 \ll \varepsilon$ must be satisfied.

We seek the solution to (1.10) as an expansion in terms of the small parameter ε. In region I

$$R=R^0+\varepsilon R^1+\ldots,\ w=w^0+\varepsilon w^1+\ldots,\ p_1=\pi^0+\varepsilon\pi^1+\ldots$$

In region II, we seek the solution in a special form:

$$R=(1-\nu)/\varepsilon+R^0+\varepsilon R^1+\ldots,\ w=w^0(\nu)+\varepsilon w^1+\ldots,\ a=\rho=1,$$

p_1 is as in region I, due to the boundary conditions.

In accordance with (1.6), (1.7), we must set $R^i = 0$ ($i = 0, 1, 2, \ldots$) for $\nu = 0$.

The solution in region II is a good approximation to real flow in tornado-like vortices outside the vortex core, as observed in both laboratory experiments [2, 3] and in nature [4].

We substitute the expansion into (1.10), keeping terms of zeroth order in ε. In region II, we obtain the equation

$$-\frac{\partial \pi^0}{\partial z}+\frac{1}{2\left(R_1^0\right)^2}\frac{\partial R_1^0}{\partial z}-g=0.$$

As already noted, π^0 and $R_1{}^0$ are the same in regions I and II. Then we can express π^0 using the last relation, and substitute this into the equations for region I. As a result, we have

$$\rho\left(\frac{\partial w^0}{\partial t}+w^0\frac{\partial w^0}{\partial z}\right)=-\frac{1-\rho_1 a_1^2}{2\left(R_1^0\right)^2}\frac{\partial R_1^0}{\partial z}+\frac{\partial}{\partial z}\int\limits_\nu^1\frac{1}{2R^0}\frac{\partial(\rho a^2)}{\partial\nu}d\nu+(1-\rho)g,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial R^0}{\partial\nu}\right)+\frac{\partial}{\partial z}\left(w^0\frac{\partial R^0}{\partial\nu}\right)=0. \tag{1.11}$$

Using $p_1 = 0$ and (1.11), it follows from (1.10) that if $\partial(\rho a^2)/\partial\nu = 0$, then this system is equivalent to the equations of a shallow-water vortex with constant density [1]. In this case, the hyperbolic nature of these systems is established in analogy to [1] by means of the appropriate change of notation.

2. The method developed is now applied to a more specific flow. A steady tornado-like vortex is examined. Let us study the flow in region I, which henceforth is called the core of the vortex. It is assumed that w = 0 in region II. In this case, system (1.10) leads to the form (1.11) directly for R and w without expansion in terms of ε. Thus, we examine (1.11) without the null superscript.

As boundary conditions, the values of w and R are prescribed at z = 0. We assume that $w = w_0(\nu)$, $R = \nu$ at z = 0. The fundamental result of separation is formulated as

THEOREM. Let a = 0, $\rho = \text{const} < 1$, $w_0 \geq \gamma > 0$, γ is a constant, and let the flow be steady, and

$$\lambda=\frac{1}{2\rho}\int\limits_0^1\frac{d\nu}{w_0^2}.$$

Then, if $\lambda < 1$, the solution exists for all $z > 0$, $\nu \in [01]$, and $R \to 0$, $w \to \infty$ monotonically for every fixed value of ν as $z \to \infty$. If $\lambda > 1$, then the solution exists only for $z \leq \ell$, where ℓ is defined by

$$l=\frac{\rho}{2(1-\rho)g}\left\{\frac{1}{\rho}-\gamma^2-\frac{1}{\rho\int\limits_0^1\frac{w_0^2\,d\nu}{\sqrt{w_0^2-\gamma^2}}}\right\},$$

where the quantity w for z = 1 approaches zero for ν, as given by the equation $w_0(\nu) = \gamma$.

Proof. We integrate (1.11) once over z. By φ we denote the quantity $\varphi = w^2 - w_0^2$. We obtain

$$\varphi-\frac{1}{\rho R_1}=\frac{2(1-\rho)}{\rho}gz-\frac{1}{\rho},$$
$$R=\int\limits_0^\nu\frac{w_0\,d\nu}{\left(w_0^2+\varphi\right)^{1/2}},\quad w=\left(w_0^2+\varphi\right)^{1/2}. \tag{2.1}$$

From the first equation it follows that $\varphi = \varphi(z)$. Thus to obtain the dependence of R and w on z, it is necessary to investigate the implicit relation $\varphi(z)$, which is given by the first equation in (2.1).

We denote $f(\varphi) = \varphi - 1/(\rho R_1)$. By differentiation we establish the relations $f'(0) = 1 - \lambda$,

$$f''(\varphi)=\frac{3}{4\rho R_1^2}\int\limits_0^1\frac{w_0 d\nu}{\left(w_0^2+\varphi\right)^{5/2}}-\frac{1}{2\rho R_1^3}\left(\int\limits_0^1\frac{w_0 d\nu}{\left(w_0^2+\varphi\right)^{3/2}}\right)^2.$$

In the calculation, it is assumed that $\varphi = 0$, $R_1 = 1$ for z = 0. Using the Bunyakovskii inequality and the expression for R_1 from the second equation in (2.1), we can write

$$\left(\int_0^1 \frac{w_0 dv}{(w_0^2+\varphi)^{3/2}}\right)^2 \leqslant R_1 \int_0^1 \frac{w_0 dv}{(w_0^2+\varphi)^{5/2}}.$$

From the latter inequality and the expression for $f''(\varphi)$ it follows that $f''(\varphi) > 0$ for $\varphi > -\gamma^2$.

Let $\lambda < 1$. Then it follows from the expression for $f'(0)$ that $f'(0) > 0$. From this and from the fact that $f''(\varphi)$ is positive, we have that $f'(\varphi) > 0$. From (2.1) we obtain $df/dz > 0$. Since $df/dz = (df/d\varphi)(d\varphi/dz)$, and, as already proved, $f'(\varphi) > 0$, then $d\varphi/dz > 0$. Thus, φ is a monotonically increasing function of z, and according to (2.1), $\varphi \to \infty$ for $z \to \infty$. Then we find from (2.1) that $w \to \infty$, $R \to 0$ monotonically as $z \to \infty$ for every fixed value of ν.

Let $\lambda > 1$. Then $f'(0) < 0$. This case is analogous to the previous one, and we find that $d\varphi/dz < 0$. Thus, φ is a monotonically decaying function. For $\varphi < -\gamma^2$, the solution does not exist, since the expression under the radical in (2.1) becomes negative. Substituting $\varphi = -\gamma^2$ into (2.1) gives the limiting value $z = \ell$, for which the solution exists. From (2.1) it also follows that for $z = \ell$, $w(\nu)$ vanishes for those ν which satisfy $w_0(\nu) = \gamma$.

3. We now apply these results to the calculation of the height of real tornado-like vortices with a warm core. Dust devils [4-6] are studied as an example of natural vortices, and experimental models [3, 7] as examples of laboratory vortices. The application of the model of a tornado-like vortex to describe real flows is based on the qualitative convergence of model and real flows, and also the fact that the outer boundary η_* does not influence the structure of (1.11).

From dust devil observations, it has been established that their cores, which are usually clearly visible due to the dust, rapidly lose their visibility at some height, becoming invisible. Frequently, before the region of disappearance, the vortex core is significantly thickened [5]. It has been shown in experiments [3, 7] that the vortex core radically changes its structure at some height, and is transformed into a convective nonrotating thermal, whose radius rapidly increases. Thus, the vortex flow being considered can change radically at a given height and can change over to another structure.

Vortices which are shed from an airplane wing were studied in [8]. We will assume, in accordance with [8], that the height of the vortex ℓ is equal to the limiting value of z up to which there exists a solution in the approximation considered here. As follows from the theorem, for this value of z, the vertical velocity vanishes for at least one point inside the core. In this way, the given assumption is analogous to the assumption that is made in boundary layer theory for the definition of the point of separation.

Calculations of vortex height were done using the model formulas and were compared with experimental data [3, 7], and with observations of dust devils as well. Subsequently all quantities and calculations are taken in dimensional form.

The upper boundary of the near-earth boundary layer (or, in the notation of [2], the lower boundary of region IV) is located in the plane $z = 0$. According to [2], the vertical velocity profile is nearly a step function. Therefore, we set $w_0 = \text{const}$ (w_0 is the dimensional vertical velocity in the vortex core at $z = 0$). The densities of the fluid in and outside the core are denoted by ρ and ρ_0, respectively, and $\rho < \rho_0$. The magnitude of the rotational component of the velocity at the edge of the core at $z = 0$ is denoted by v_0. Then the condition that the vortex height $\lambda > 1$ be bounded and the expression for ℓ take on the forms

$$\frac{2\rho w_0^2}{\rho_0 v_0^2} < 1,\ l = \frac{\rho v_0^2}{2(\rho_0-\rho)g}\left(\frac{\rho_0}{\rho} - \frac{w_0^2}{v_0^2}\right).$$

From these relations it follows that ℓ depends weakly on w_0, and therefore in terms of order of magnitude, we write

$$l \approx \rho_0 v_0^2/[2g(\rho_0-\rho)].$$

For $(\rho_0 - \rho)/\rho_0$, the estimate $(\rho_0 - \rho)/\rho_0 \approx (T - T_0)/T_0$ is valid, if $(T - T_0) \ll T_0$ (T is the mean temperature in the core, and T_0 is the temperature of the surrounding air). Then we obtain

$$l \approx v_0^2 T_0/(2g(T - T_0)).$$

We set v_0 = 100 cm/sec, w_0 = 43 cm/sec, $T - T_0$ = 20 K, T_0 = 300 K [7]. The value of v_0 is computed from the condition of conservation of circulation of the velocity about the core in the outer region, and so $2w_0^2/v_0^2 < 1$, that is, $\lambda > 1$, $\ell \approx 85$ cm. According to measurements in [7], $\ell \approx 45$ cm.

Setting v_0 = 40 cm/sec, $T - T_0$ = 10 K, T_0 = 300 K [3], we obtain $\ell \approx 24$ cm. According to [3], $\ell \approx 60$ cm.

For dust devils we take v_0 = 10 m/sec, $T - T_0$ = 2 K, T_0 = 300 K [4], for which $\ell \approx$ 750 m. The height of the vortex is approximately 600 m in the photograph shown in [5].

Thus, the theoretical results are qualitatively supported by observational data. Quantitative calculation of vortex height using the model formulas agrees in order of magnitude with the results of laboratory measurements and observations of naturally occurring dust devils.

LITERATURE CITED

1. V. M. Teshukov, "Hyperbolicity of the equations of long waves," Dokl. Akad. Nauk SSSR, 284, No. 3 (1985).
2. V. V. Nikulin, "Investigation of the interaction of tornado-like vortices and rigid boundaries," Prikl. Mekh. Tekh. Fiz., No. 1 (1980).
3. J. B. Mullen and T. Maxworthy, "A laboratory model of dust devil vortices," Dyn. Atmos. Oceans, No. 1 (1977).
4. P. C. Sinclair, "The lower structure of dust devils," J. Atmos. Sci., 30, No. 8 (1973).
5. R. L. Ives, "Behavior of a dust devil," Bull. Am. Meteorol. Soc., 28, No. 4 (1947).
6. N. R. Williams, "Development of dust whirls and similar small-scale vortices," Bull. Am. Meteorol. Soc., 29, No. 3 (1948).
7. D. E. Fitzjerald, "A laboratory simulation of convective vortices," J. Atmos. Sci., Vol. 30, No. 7 (1973).
8. A. Mager, "Dissipation and breakdown of a wing-tip vortex," J. Fluid Mech., 55, No. 4 (1972).